Math 102

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November 13, 2018

Announcements

- Midterm regrade forms - please submit any regrade requests by tomorrow! You can give them to me OR give them directly to Elyse Yeager (Math 229).
- Reminder: The final exam is in the afternoon on **Tuesday December 4.** Mark your calendar if you haven't done so already.

Goals Today

- Different types of differential equations
- Using slope fields
- More practice

Recall...

$$y' = 4 - 3y \qquad y' = 2t \qquad y' = yt^2$$

$$2yy'' + y' - 3t = 0 \qquad y'' = 3y$$

- A differential equation is an equation which relates the behavior of a function $y(t)$ to its derivative, $\frac{dy}{dt}$ (also written y').

Recall...

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$$2yy'' + y' - 3t = 0 \qquad y'' = 3y$$

- A differential equation is an equation which relates the behavior of a function $y(t)$ to its derivative, $\frac{dy}{dt}$ (also written y').
- A **solution** is a function $y(t)$ which satisfies the stated equation.

$$y' = 4 - 3y \qquad y' = 2t \qquad y' = yt^2$$

$$2yy'' + y' - 3t = 0 \qquad y'' = 3y$$

- Differential equations can also involve the second derivative, $\frac{d^2y}{dt^2}$ (also written y''). Sometimes they can even involve higher derivatives.

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- Differential equations can also involve the second derivative, $\frac{d^2y}{dt^2}$ (also written y''). Sometimes they can even involve higher derivatives.
- A differential equation is called **first order** if it involves only y, y', and t. (no second derivatives or higher)

Some Real-life Examples

Thrown ball	$y'' = -9.8$
Cooling/Heating	$T' = 20 - T$
Drug metabolism	$D' = 30 - 5D$
Population growth	$P' = P(20 - P)$
Spread of Disease	$I' = 2I(500 - I)$
Oscillating Spring	$x'' = -3x$
Swinging pendulum	$\theta'' = -\sin(\theta)$

- An autonomous first-order DE is a DE of the form

$$y' = f(y)$$

- A linear first-order DE is an autonomous DE of the form

$$y' + ay + b = 0$$

Exercise: Sort the DE's!

$$y' = 4 - 3y \qquad y' = 2t \qquad y' = yt^2$$

$$2yy'' + y' - 3t = 0 \qquad y'' = 3y$$

	First-order	Higher-order
Autonomous		
Not Autonomous		

Exercise: Sort the DE's!

$$y' = 4 - 3y \qquad y' = 2t \qquad y' = yt^2$$

$$2yy'' + y' - 3t = 0 \qquad y'' = 3y$$

	First-order	Higher-order
Autonomous	$y' = 4 - 3y$	$y'' = 3y$
Not Autonomous	$y' = 2t$ $y' = yt^2$	$2yy'' + y' - 3t = 0$

Autonomous and Linear DE's

- A linear first-order DE is an autonomous DE of the form $y' + ay + b = 0$. We can **solve** these, by substitution.
- An autonomous first-order DE is a DE of the form $y' = f(y)$. We have not solved these, but can analyze these with **qualitative analysis**.

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- More general DE's can't be solved explicitly. But we still have tools we can use
  - Qualitative methods (Slope fields)
  - Approximation methods (Euler's method)

$$y' = t/y$$

Suppose that $y(t)$ is a function which satisfies the differential equation above. Suppose that $y(2) = 3$. Then what is $y'(2)$?

Slope Fields

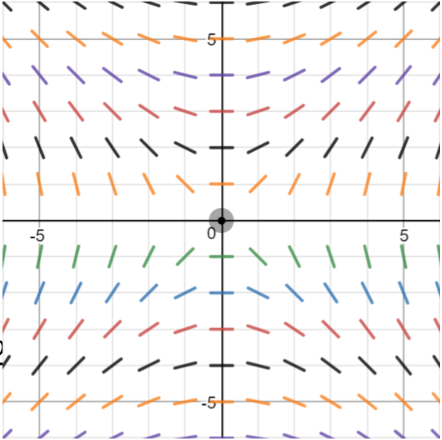
$$y' = t/y$$

$$(2,2) \implies y' = 1$$

$$(3,1) \implies y' = 3$$

$$(1,3) \implies y' = 1/3$$

$$(-5,2) \implies y' = -5/2$$

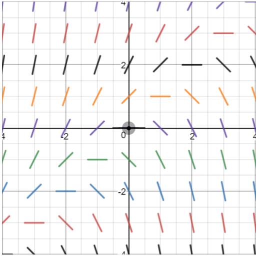
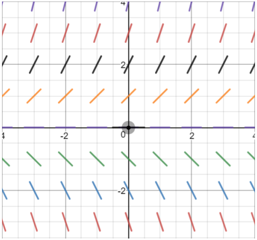
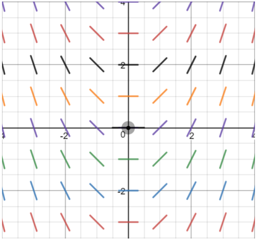


Definition: A **slope field** is a pictorial representation of y' at every point in the (t,y)-plane.

Slope Fields

$$y' = y \qquad\qquad y' = t \qquad\qquad y' = y - t$$

Question: Match the DE's above to the slope fields below.

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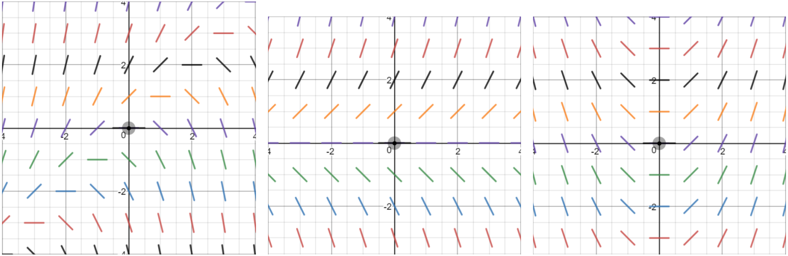
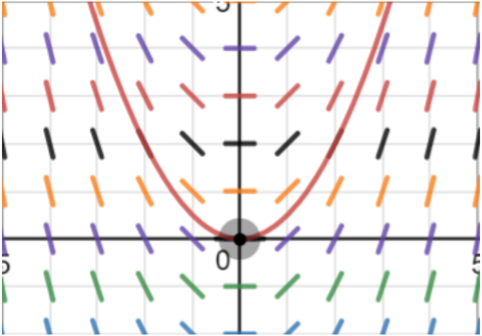


$$y' = y - t \qquad y' = y \qquad y' = t$$

Slope Fields

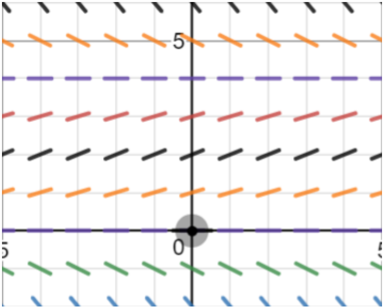
Pictured: The slope field of $y' = t$ and the graph of $y = \frac{t^2}{2}$.



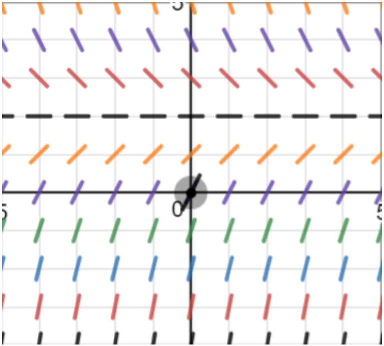
A solution to the differential equation must follow the slope field.

Slope Field of an Autonomous DE

$y' = y(4-y)$



$y' = 2 - y$



Question: What can you say about the slope field of an autonomous differential equation?

Slope Field of an Autonomous DE

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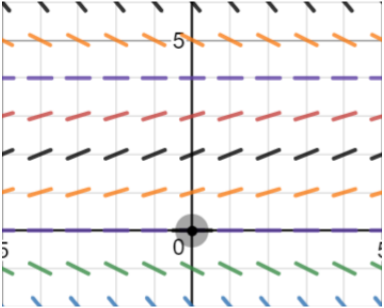

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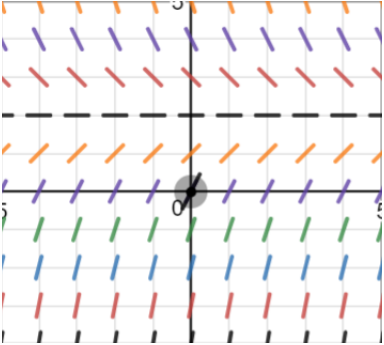

Question: What can you say about the slope field of an autonomous differential equation? Along any horizontal line, the slope lines have the same slope.

Some weirdness that can happen

Question: Consider the autonomous differential equation

$$y' = \sqrt{y+1}$$

It would seem that $y(t) = \frac{1}{4}t^2 - 1$ satisfies this differential equation. Is $y(t) = \frac{1}{4}t^2 - 1$ really a solution?

Demonstration: Desmos and dfield.

More Practice - adapted from 2017 Final Exam Q8

Question: An egg was laid ten minutes ago. When the egg was laid, it was at 40 degrees. The current temperature of the egg is 25 degrees. The ambient temperature is 10 degrees. What temperature will the egg be in 10 minutes?

Question: How long will it take until the egg has a temperature of 14 degrees? You do not have to simplify your answer.

More Practice - adapted from 2017 Final Exam Q8

Question: An egg was laid ten minutes ago. When the egg was laid, it was at 40 degrees. The current temperature of the egg is 25 degrees. The ambient temperature is 10 degrees. What temperature will the egg be in 10 minutes? 17.5 degrees

Question: How long will it take until the egg has a temperature of 14 degrees? You do not have to simplify your answer. $10 \cdot \log_2(15/4)$.

More Practice

Question: A cup of just-boiled tea (100 C) is placed outside. After ten minutes, it is at 40 C, and after twenty minutes, it is at 25 C. What is the temperature outside?

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$$\frac{100 - E}{40 - E} = \frac{40 - E}{25 - E}$$

$$(100 - E)(25 - E) = (40 - E)(40 - E)$$

$$2500 - 125E + E^2 = 1600 - 80E + E^2$$

$$900 = 45E$$

$$\boxed{E = 20}$$

Extra Practice

Question: A barrel initially contains 2 kg of salt dissolved in 20 L of water. Water flows in at a rate of 0.5 L/min, and well-mixed salt water solution flows out at the same rate.

- Write down a differential equation for $S(t)$, the amount of salt in the barrel at time t.
- How many minutes will it take before there is only 1 kg of salt left in the barrel?